8 Linear systems in $\mathbf{R}^d$

8.1 General theory

In the previous lecture I discussed a few things about planar linear autonomous ODE. However, most of this discussion can be extended to d-dimensional space $\mathbf{R}^d$ without much of a change. Especially simple are the formulations in the *generic* case, and I will stick to it to give an idea what can be expected in such systems.

I consider

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x}(t) \in \mathbf{R}^d,\ d \geq 1, \tag{1}$$

with the initial condition

$$\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbf{R}^d. \tag{2}$$

The unique solution to (1)–(2) is given by the same formula

$$\boldsymbol{x}(t; \boldsymbol{x}_0) = e^{\boldsymbol{A}t}\boldsymbol{x}_0,$$

where the matrix exponent is defined in the previous lecture.

I assume that matrix $\boldsymbol{A}$ is *hyperbolic.* To wit, let d be the number of eigenvalues of $\boldsymbol{A}$ counting multiplicities, and let d_-, d_+, and d_0 denote the number of eigenvalues with negative, positive and zero real parts respectively. I have $d_0 + d_- + d_+ = d$.

Definition 1. *System* (1)*, as well as matrix* $\boldsymbol{A}$*, as well as the equilibrium* $\hat{\boldsymbol{x}} = \boldsymbol{0} \in \mathbf{R}^d$*, are called hyperbolic if* $d_0 = 0$*. Moreover,* $\hat{\boldsymbol{x}}$ *is called a hyperbolic saddle if* $d_+ d_- \neq 0$*.*

To be hyperbolic is a *generic* property: almost all the matrices are hyperbolic. Moreover, a hyperbolic system (1) can have only one equilibrium $\hat{\boldsymbol{x}}$ because in this case I have that $\det \boldsymbol{A} \neq 0$ (recall that I have $\det \boldsymbol{A} = \prod_{i=1}^{d} \lambda_i$).

Another generic property is to have distinct eigenvalues (i.e., there are no multiple eigenvalues). In this case I know from the linear algebra that the list of eigenvectors $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d)$ corresponding to the eigenvalues is linearly independent and hence forms a basis of $\mathbf{R}^d$ (if matrix $\boldsymbol{A}$ is real then I always have the basis of real vectors in the sense that for the complex conjugate pair of eigenvalues λ_i and $\lambda_{j+1} = \overline{\lambda}_j$ I take $\operatorname{Re} \boldsymbol{v}_j$ and $\operatorname{Im} \boldsymbol{v}_j$ as the real basis vectors). Let me denote T_-, T_+, T_0 the subspaces formed by the spans of the vectors corresponding to the eigenvalues with negative real part, positive real part, and zero real part respectively. They are called *stable, unstable, and neutral subspaces* respectively. If all the eigenvalues are distinct then

$$\mathbf{R}^d = T_- \oplus T_+ \oplus T_0,$$

i.e., any element of $\boldsymbol{v} \in \mathbf{R}^d$ can be *uniquely* represented

$$\boldsymbol{v} = \boldsymbol{v}_- + \boldsymbol{v}_+ + \boldsymbol{v}_0,$$

where $\boldsymbol{v}_- \in T_-$, $\boldsymbol{v}_+ \in T_+$, $\boldsymbol{v}_0 \in T_0$.

An important property of $T_\pm, T_0$ is their *invariance* with respect to the flow defined by (1). This means that if the initial condition, e.g., $\boldsymbol{x}_0 \in T_-$ then $\boldsymbol{x}(t; \boldsymbol{x}_0) \in T_-$ for any $t \to \pm\infty$.

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Proposition 2. *Subspaces $T_\pm, T_0$ are invariant with respect to the flow of* (1).

Proof. Consider, e.g., $\boldsymbol{x}_0 \in T_-$. It means that

$$\boldsymbol{x}_0 = C_1\boldsymbol{v}_1 + \ldots + C_{d_-}\boldsymbol{v}_{d_-},$$

such that for each $\boldsymbol{v}_i$ there is λ_i such that $\operatorname{Re}\lambda_i < 0$. Consider vector-function

$$\boldsymbol{\phi}(t) = C_1 e^{\lambda_1 t}\boldsymbol{v}_1 + \ldots + C_{d_-}e^{\lambda_{d_-}t}\boldsymbol{v}_{d_-}.$$

I have $\boldsymbol{\phi}(0) = \boldsymbol{x}_0$ and this function satisfies (1) (check it). Therefore $\boldsymbol{x}(t;\boldsymbol{x}_0) = \boldsymbol{\phi}(t)$ is the unique solution, and by construction $\boldsymbol{\phi}(t) \in T_-$ for any t. ■

The general theory discussed so far implies

Theorem 3. *Let $\mathbf{R}^d = T_- \oplus T_+ \oplus T_0$ and $\det \boldsymbol{A} \neq 0$. Then the unique equilibrium $\hat{\boldsymbol{x}}$ is Lyapunov stable if and only if $T_+ = \emptyset$. $\hat{\boldsymbol{u}}$ is asymptotically stable if and only if $T_0 = T_+ = \emptyset$.*

The condition $\mathbf{R}^d = T_- \oplus T_+ \oplus T_0$ means that I have exactly d linearly independent eigenvectors of $\boldsymbol{A}$. This is not always true in the case when there are eigenvalues of multiplicities larger than one. But even in this case I have

Theorem 4. *Consider* (1). *If for all eigenvalues of $\boldsymbol{A}$ I have $\operatorname{Re}\lambda_i < 0$ then the unique equilibrium $\hat{\boldsymbol{x}}$ is asymptotically stable.*

8.1.1 Examples of the phase portraits in $\mathbf{R}^3$

Consider several examples of three dimensional hyperbolic equilibria and their phase portraits.

The first example is for

$$\boldsymbol{A}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Here I have the eigenvalues $-1, -2, 1$ and the corresponding eigenvectors coincide with the coordinate axes. According to the general theory I have a stable subspace T_- spanned by two standard coordinate vectors $\boldsymbol{e}_1 = (1,0,0)$ and $\boldsymbol{e}_2 = (0,1,0)$, whereas the subspace spanned by $\boldsymbol{e}_3 = (0,0,1)$ is unstable (see the figure).

The second example is for

$$\boldsymbol{A}_2 = \begin{bmatrix} 1/2 & 2 & 0 \\ 8 & 1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

and hence the eigenvalues are $1/2 \pm 4\mathrm{i}$ and -1. The two eigenvectors corresponding to the complex conjugate eigenvalues can be used to form a two dimensional real subspace T_+, which is unstable. The vector $\boldsymbol{e}_3$ spans the stable subspace T_-.

For the third example I picked

$$\boldsymbol{A}_3 = \begin{bmatrix} -1/2 & 2 & 0 \\ 8 & -1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

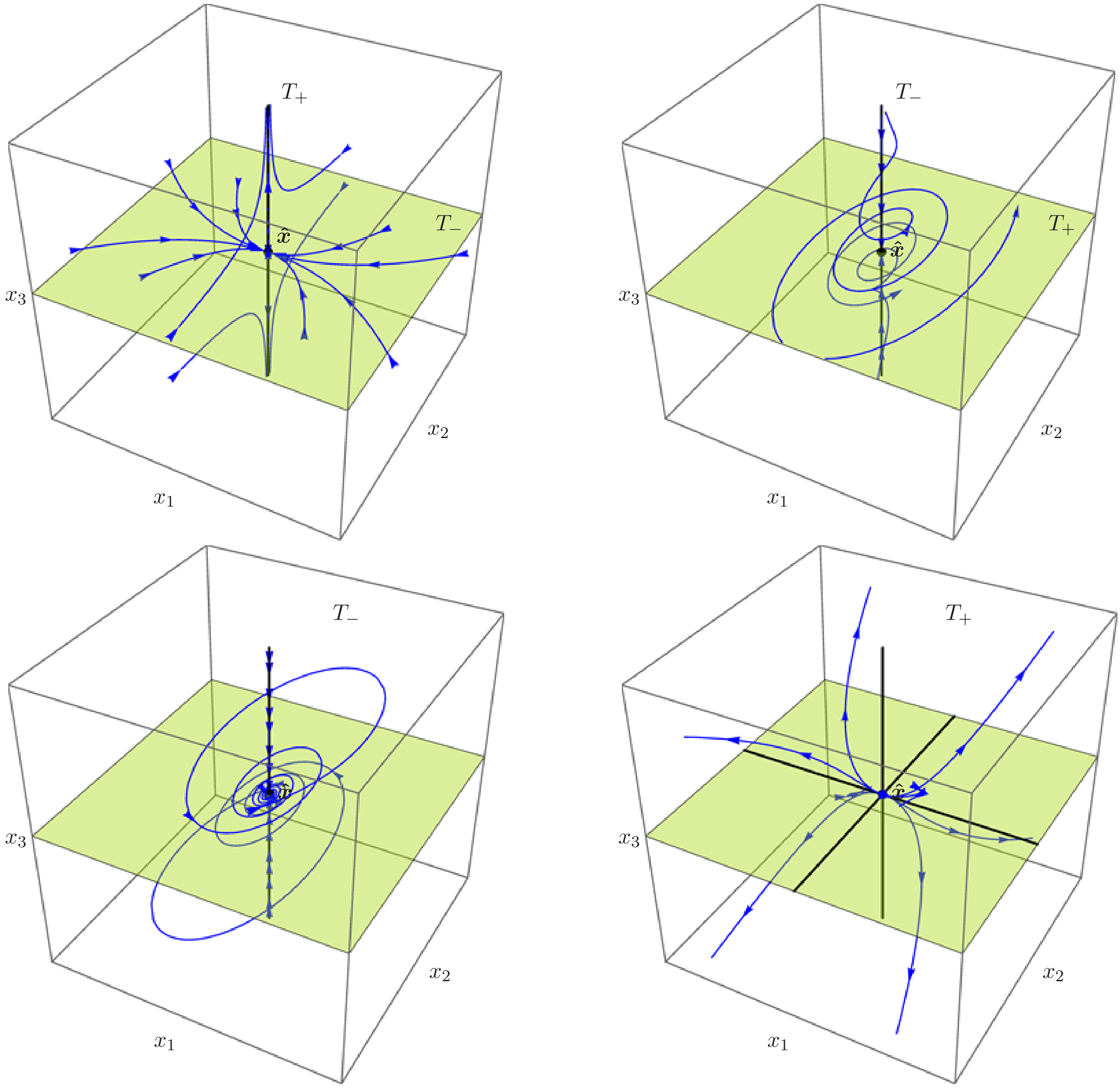

Figure 1: Three dimensional phase portraits. A hyperbolic saddle with two dimensional stable subspace T_- and one dimensional unstable subspace T_+ for matrix $\boldsymbol{A}_1$ (see the text for details). A hyperbolic saddle with one dimensional stable subspace T_- and two dimensional unstable subspace T_+ for $\boldsymbol{A}_2$. A hyperbolic sink with three dimensional stable subspace $T_- = \mathbf{R}^3$ for $\boldsymbol{A}_3$. A hyperbolic source with three dimensional unstable subspace $T_+ = \mathbf{R}^3$ for $\boldsymbol{A}_4$

and hence all three eigenvalues have negative real parts. Therefore in this case my state space $\mathbf{R}^3$ coincides with the stable subspace T_-.

Finally, take

$$\boldsymbol{A}_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

with the real negative eigenvalues. The phase space here coincides with the unstable subspace T_+.

8.1.2 Routh–Hurwitz criteria

The asymptotic stability of the equilibrium point of the linear system is determined by the condition $\operatorname{Re}\lambda_i < 0$, where λ_i are the roots of

$$P(\lambda) = \lambda^d + a_1\lambda^{d-1} + \ldots + a_{d-1}\lambda + a_d.$$

Therefore it is of great use to have a condition which, without explicit calculations of the roots, would provide us with information on the signs on the real parts of the roots. One of such conditions, and arguably most used, is the Routh–Hurwitz criterion. I just formulate it here, proofs can be found elsewhere[1].

Consider a sequence of matrices

$$\boldsymbol{H}_1 = a_1, \quad \boldsymbol{H}_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}, \quad \boldsymbol{H}_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}, \ldots$$

$$\boldsymbol{H}_d = \begin{bmatrix} a_1 & 1 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & \ldots & 0 \\ a_5 & a_4 & a_3 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & a_d \end{bmatrix},$$

where $\boldsymbol{H}_j$ are the main corner minors of the last matrix $\boldsymbol{H}_d$. $\boldsymbol{H}_d$ is written as follows. First I put on the main diagonal the coefficients from a_1 to a_d. After this I fill the columns such that the column with odd index can have only odd coefficients, and the columns with even indexes have only even coefficients. I put 1 for a_0 and zero for any coefficients a_k for $k < 0$ and $k > d$.

Theorem 5 (Routh–Hurwitz). *For all the roots of the characteristic polynomial $P(\lambda)$ to have negative real parts it is necessary and enough that*

$$\det \boldsymbol{H}_i > 0, \quad i = 1, \ldots, d.$$

Corollary 6. *For $d = 2, 3, 4$ the necessary and sufficient conditions for the characteristic polynomial to have all the roots with negative real part can be written as follows:*

$$d = 2\colon\ a_1 > 0,\ a_2 > 0.$$
$$d = 3\colon\ a_1 > 0, a_3 > 0, a_1a_2 > a_3.$$
$$d = 4\colon\ a_1 > 0, a_3 > 0, a_4 > 0, a_1a_2a_3 > a_3^2 + a_1^2a_4.$$

Corollary 7. *The following necessary condition is true: If all the roots of the characteristic polynomial have negative real parts then*

$$a_j > 0, \quad j = 1, \ldots, d.$$

[1]e.g., Gantmacher, F. R., & Brenner, J. L. (2005). Applications of the Theory of Matrices. Courier Corporation.